

ECE4893A/CS4803MPG:

MULTICORE AND GPU PROGRAMMING FOR VIDEO GAMES

Supplemental material: Quaternions



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Quaternion definition

$$\mathbf{q} = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \equiv (w, x, y, z) \\ \equiv (w, \mathbf{v}) \\ \mathbf{v} = (x, y, z)$$

$$\mathbf{i}\mathbf{i} = -1 \quad \mathbf{j}\mathbf{j} = -1 \quad \mathbf{k}\mathbf{k} = -1 \quad \mathbf{i}\mathbf{j}\mathbf{k} = -1$$

$$\mathbf{i}\mathbf{j} = \mathbf{k} \quad \mathbf{j}\mathbf{i} = -\mathbf{k}$$

$$\mathbf{j}\mathbf{k} = \mathbf{i} \quad \mathbf{k}\mathbf{j} = -\mathbf{i}$$

$$\mathbf{k}\mathbf{i} = \mathbf{j} \quad \mathbf{i}\mathbf{k} = -\mathbf{j}$$

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Quaternion multiplication

$$\mathbf{q} = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (w, \mathbf{v}) \\ \mathbf{v} = (x, y, z)$$

$$\mathbf{q}_1\mathbf{q}_2 = (w_1w_2 - \mathbf{v}_1 \cdot \mathbf{v}_2, w_1\mathbf{v}_2 + w_2\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2) \\ = (w_1w_2 - x_1x_2 - y_1y_2 - z_1z_2, \\ w_1x_2 + x_1w_2 + y_1z_2 - z_1y_2, \\ w_1y_2 + y_1w_2 + z_1x_2 - x_1z_2, \\ w_1z_2 + z_1w_2 + x_1y_2 - y_1x_2)$$

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Quaternion conjugate and inverse

$$\text{Conjugate: } \mathbf{q}^* = w - x\mathbf{i} - y\mathbf{j} - z\mathbf{k} = (w, -\mathbf{v})$$

$$\text{Multiplicative inverse: } \mathbf{q}^{-1} = \frac{\mathbf{q}^*}{\|\mathbf{q}\|}$$

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Quaternion representations of rotations

- Quaternion representation of a rotation of θ radians around axis defined by the unit vector $\bar{\mathbf{u}}$

$$\mathbf{q} = \left(\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\bar{\mathbf{u}} \right)$$

- Notice \mathbf{q} has unit norm
- Rotations compose with quaternion multiplication

Rotating points

- Embed 3-D Cartesian coordinates in last three components of a quaternion

$$\begin{aligned} (0, x', y', z') &= \mathbf{q}(0, x, y, z)\mathbf{q}^{-1} \\ &= \mathbf{q}(0, x, y, z)\mathbf{q}^* \end{aligned}$$

Unit quaternion to rotation matrix

$$\mathbf{R} = \begin{bmatrix} 1 - 2y^2 - 2z^2 & 2xy - 2wz & 2xz + 2wy \\ 2xy + 2wz & 1 - 2x^2 - 2z^2 & 2yz - 2wx \\ 2xz - 2wy & 2yz + 2wx & 1 - 2x^2 - 2y^2 \end{bmatrix}$$

- Applying matrix to a vector takes a few operations less than applying quaternion to a vector

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \mathbf{R} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$