Time Averaging

ECE 6279: Spatial Array Processing
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Lecture 14

Prof. Aaron D. Lanterman

School of Electrical & Computer Engineering
Georgia Institute of Technology
AL: 404-385-2548
<lanterma@ece.gatech.edu>

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Where We Are (and Aren't) in J&D

- Inspired by Section 4.9.1

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Average Beamformer Output

- Average power of beamformer output over time

$$\hat{z} = \frac{1}{L}\sum_{l=0}^{L-1}\left|z(l)\right|^2 = \frac{1}{L}\sum_{l=0}^{L-1}\left|\mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{y}(l)\right|^2$$
$$= \frac{1}{L}\sum_{l=0}^{L-1}\mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{y}(l)\mathbf{y}^H(l)\mathbf{W}^H\mathbf{e}(\vec{k})$$
$$= \mathbf{e}^H(\vec{k})\mathbf{W}\left[\frac{1}{L}\sum_{l=0}^{L-1}\mathbf{y}(l)\mathbf{y}^H(l)\right]\mathbf{W}^H\mathbf{e}(\vec{k})$$

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Under the Stochastic Model

$$\hat{z} = \mathbf{e}^H(\vec{k})\mathbf{W}\hat{\mathbf{R}}_{\mathbf{y}}\mathbf{W}^H\mathbf{e}(\vec{k})$$

where $\hat{\mathbf{R}}_{\mathbf{y}} = \frac{1}{L}\sum_{l=0}^{L-1}\mathbf{y}(l)\mathbf{y}^H(l)$

- Under the stochastic signal model, $\hat{\mathbf{R}}_{\mathbf{y}}$ is the unconstrained maximum-likelihood estimate of the correlation matrix of a Gaussian random vector $\underline{\mathbf{y}}$

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Improvement from Time Averaging

$$\hat{\mathbf{R}}_{\mathbf{y}} \rightarrow \mathbf{R}_{\mathbf{y}} = E[\underline{\mathbf{y}}\underline{\mathbf{y}}^T] \text{ as } L \rightarrow \infty$$

- If snapshots are independent, variance of estimate improves as 1/*L*, standard deviation improves as 1/sqrt(*L*)
- Can't make *L* too big since the sources are moving!

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Outer Product of Steering Vectors

- For linear array

$$\gamma = k\sin(\phi)\cos(\theta)d$$
$$= \frac{2\pi}{\lambda}\sin(\phi)\cos(\theta)d$$

$$\begin{bmatrix} 1 \\ e^{j\gamma} \\ \vdots \\ e^{j(M-1)\gamma} \end{bmatrix} \begin{bmatrix} 1 & e^{-j\gamma} & \cdots & e^{-j(M-1)\gamma} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & e^{-j\gamma} & \cdots & e^{-j(M-1)\gamma} \\ e^{j\gamma} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & e^{-j\gamma} \\ e^{j(M-1)\gamma} & \cdots & e^{j\gamma} & 1 \end{bmatrix}$$

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Linear Array, Single Plane Wave

- Assuming no noise

$$\mathbf{y}(l) = s(l)\exp\left(-j\frac{M-1}{2}\gamma\right)\exp\begin{bmatrix} 0 \\ j\gamma \\ \vdots \\ j(M-1)\gamma \end{bmatrix}$$

$$\mathbf{y}(l)\mathbf{y}^H(l) =$$

$$|s(l)|^2 \exp\begin{bmatrix} 0 & -j\gamma & \cdots & -j(M-1)\gamma \\ j\gamma & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -j\gamma \\ j(M-1)\gamma & \cdots & j\gamma & 0 \end{bmatrix}$$

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Toeplitz Structure of Est. Correlation

$$\hat{\mathbf{R}}_{\mathbf{y}} = \frac{1}{L}\sum_{l=0}^{L-1}\mathbf{y}(l)\mathbf{y}^H(l)$$

- $\hat{\mathbf{R}}_{\mathbf{y}}$ is Toeplitz since a sum of Toeplitz matrices is Toeplitz
- Remember assuming no noise

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Convergence to True Correlation

$$\hat{\mathbf{R}}_{\mathbf{y}} \to E\left[|\underline{s}|^2\right] \exp\begin{bmatrix} 0 & -j\gamma & \cdots & -j(M-1)\gamma \\ j\gamma & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -j\gamma \\ j(M-1)\gamma & \cdots & j\gamma & 0 \end{bmatrix}$$

as $L \to \infty$

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Linear Array, Two Plane Waves

- **Again assuming no noise**

$$\mathbf{y}(l) = s_1(l)\exp\left(-j\frac{M-1}{2}\gamma_1\right)\exp\begin{bmatrix} 0 \\ j\gamma_1 \\ \vdots \\ j(M-1)\gamma_1 \end{bmatrix}$$

$$+s_2(l)\exp\left(-j\frac{M-1}{2}\gamma_2\right)\exp\begin{bmatrix} 0 \\ j\gamma_2 \\ \vdots \\ j(M-1)\gamma_2 \end{bmatrix}$$

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Two Signals

$$\mathbf{y}(l)\mathbf{y}^H(l) = |s_1(l)|^2 \begin{bmatrix} 1 & e^{-j\gamma_1} & \cdots & e^{-j(M-1)\gamma_1} \\ e^{j\gamma_1} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & e^{-j\gamma_1} \\ e^{j(M-1)\gamma_1} & \cdots & e^{j\gamma_1} & 1 \end{bmatrix}$$

$$+|s_2(l)|^2\left[\text{same but with } \gamma_2\right]$$

$$+\text{crossterms!!!}$$

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Crossterms (1)

$$s_1(l)e^{-j\frac{M-1}{2}\gamma_1}\begin{bmatrix} 1 \\ e^{j\gamma_1} \\ \vdots \\ e^{j(M-1)\gamma_1} \end{bmatrix} s_2^*(l)e^{j\frac{M-1}{2}\gamma_2}\begin{bmatrix} 1 \\ e^{-j\gamma_2} \\ \vdots \\ e^{-j(M-1)\gamma_2} \end{bmatrix}^T$$

$$+s_2(l)e^{-j\frac{M-1}{2}\gamma_2}\begin{bmatrix} 1 \\ e^{j\gamma_2} \\ \vdots \\ e^{j(M-1)\gamma_2} \end{bmatrix} s_1^*(l)e^{j\frac{M-1}{2}\gamma_1}\begin{bmatrix} 1 \\ e^{-j\gamma_1} \\ \vdots \\ e^{-j(M-1)\gamma_1} \end{bmatrix}^T$$

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Crossterms (2)

$$s_1(l)s_2^*(l)e^{-j\frac{(M-1)}{2}(\gamma_1-\gamma_2)}\times$$

$$\exp\begin{bmatrix} 0 & -j\gamma_2 & \cdots & -j(M-1)\gamma_2 \\ j\gamma_1 & j(\gamma_1-\gamma_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & j[(M-2)\gamma_1-(M-2)\gamma_2] \\ j(M-1)\gamma_1 & \cdots & j[(M-1)\gamma_1-(M-2)\gamma_2] & j(M-1)(\gamma_1-\gamma_2) \end{bmatrix}$$

$$+s_2(l)s_1^*(l)e^{-j\frac{(M-1)}{2}(\gamma_2-\gamma_1)}\times$$

$$\exp\begin{bmatrix} 0 & -j\gamma_1 & \cdots & -j(M-1)\gamma_1 \\ j\gamma_2 & j(\gamma_2-\gamma_1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & j[(M-2)\gamma_2-(M-2)\gamma_1] \\ j(M-1)\gamma_2 & \cdots & j[(M-1)\gamma_2-(M-2)\gamma_1] & j(M-1)(\gamma_2-\gamma_1) \end{bmatrix}$$

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Crossterm Matrix Elements

- ***i*th diagonal element of crossterm matrix given by**

$$2\mathrm{Re}\left\{s_1(l)s_2^*(l)\exp\left(-j\frac{M-1}{2}[\gamma_1-\gamma_2]\right)\exp\left(j[\gamma_1-\gamma_2]i\right)\right\}$$

$$=2\mathrm{Re}\left\{s_1(l)s_2^*(l)\right\}\cos\left([\gamma_1-\gamma_2]\left[i-\frac{M-1}{2}\right]\right)$$

according to J&D, but Aaron is not sure he buys that

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Aaron's Calculatiion

- ***i*th diagonal element of crossterm matrix given by**

$$2\mathrm{Re}\left\{s_1(l)s_2^*(l)\exp\left(-j\frac{M-1}{2}[\gamma_1-\gamma_2]\right)\exp\left(j[\gamma_1-\gamma_2]i\right)\right\}$$

$$=2\mathrm{Re}\left\{s_1(l)s_2^*(l)\right\}\cos\left([\gamma_1-\gamma_2]\left[i-\frac{M-1}{2}\right]\right)$$

$$-2\mathrm{Im}\left\{s_1(l)s_2^*(l)\right\}\sin(\cdots)$$

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Intersignal Coherence

$$\frac{1}{L}\sum_{l=0}^{L-1}\mathrm{Re}\left\{s_1(l)s_2^*(l)\right\}\rightarrow\underbrace{\mathrm{Re}\left\{E[\underline{s}_1\underline{s}_2^*]\right\}}_{\text{Intersignal coherence}}\text{ as } L\rightarrow\infty$$

- **Incoherent signals have an intersignal coherence of zero**
- **Coherence suggests multipath effects**

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