ECE4893A/CS4803MPG:

Multicore and GPU Programming for Video Games

Supplemental material: Quaternions

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Quaternion definition

$$\mathbf{q} = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \equiv (w, x, y, z)$$

$$\equiv (w, \mathbf{v})$$

$$\mathbf{v} = (x, y, z)$$

$$\mathbf{ii} = -1 \quad \mathbf{jj} = -1 \quad \mathbf{kk} = -1 \quad \mathbf{ijk} = -1$$

$$\mathbf{ij} = \mathbf{k} \quad \mathbf{ji} = -\mathbf{k}$$

$$\mathbf{jk} = \mathbf{i} \quad \mathbf{kj} = -\mathbf{i}$$

$$\mathbf{ki} = \mathbf{j} \quad \mathbf{ik} = -\mathbf{j}$$

Quaternion multiplication

$$\mathbf{q} = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (w, \mathbf{v})$$

$$\mathbf{v} = (x, y, z)$$

$$\mathbf{q}_1\mathbf{q}_2 = (w_1 w_2 - \mathbf{v}_1 \cdot \mathbf{v}_2, w_1\mathbf{v}_2 + w_2\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2)$$

$$\begin{aligned}
= (&w_1 w_2 - x_1 x_2 - y_1 y_2 - z_1 z_2,\\
&w_1 x_2 + x_1 w_2 + y_1 z_2 - z_1 y_2,\\
&w_1 y_2 + y_1 w_2 + z_1 x_2 - x_1 z_2,\\
&w_1 z_2 + z_1 w_2 + x_1 y_2 - y_1 x_2)
\end{aligned}$$

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Quaternion conjugate and inverse

Conjugate: $\mathbf{q}^* = w - x\mathbf{i} - y\mathbf{j} - z\mathbf{k} = (w, -\mathbf{v})$

Multiplicative inverse: $\mathbf{q}^{-1} = \dfrac{\mathbf{q}^*}{\|\mathbf{q}\|}$

Quaternion representations of rotations

- Quaternion representation of a rotation of θ radians around axis defined by the unit vector $\vec{\mathbf{u}}$

$$\mathbf{q} = \left(\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\vec{\mathbf{u}} \right)$$

- Notice $\mathbf{q}$ has unit norm
- Rotations compose with quaternion multiplication

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Rotating points

- Embed 3-D Cartesian coordinates in last three components of a quaternion

$$
\begin{aligned}
(0, x', y', z') &= \mathbf{q}(0, x, y, z)\mathbf{q}^{-1} \\
&= \mathbf{q}(0, x, y, z)\mathbf{q}^{*}
\end{aligned}
$$

Unit quaternion to rotation matrix

$$\mathbf{R} = \begin{bmatrix} 1-2y^2-2z^2 & 2xy-2wz & 2xz+2wy \\ 2xy+2wz & 1-2x^2-2z^2 & 2yz-2wx \\ 2xz-2wy & 2yz+2wx & 1-2x^2-2y^2 \end{bmatrix}$$

- Applying matrix to a vector takes a few operations less than applying quaternion to a vector

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Advice from Simon Brown, “Representing Rotations in Quaternion Arithmetic”
www.sjbrown.co.uk/?article=quaternions