

## Unconstrained Covariance Estimation

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Lecture 33

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## Complex Gaussian Data

$$\underline{\mathbf{y}} \sim \mathcal{CN}(0, \mathbf{K}_y)$$

- Likelihood density:  $p(\mathbf{y}) =$

$$\frac{1}{[\pi^M \det \mathbf{K}_y]^L} \exp\left[-\sum_{l=0}^{L-1} \mathbf{y}^H(l) \mathbf{K}_y^{-1} \mathbf{y}(l)\right]$$

- Loglikelihood:

$$-L \ln \det \mathbf{K}_y - L \text{tr}\left[\hat{\mathbf{K}}_y \mathbf{K}_y^{-1}\right]$$

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## References

- [en.wikipedia.org/wiki/  
Estimation\\_of\\_covariance\\_matrices](http://en.wikipedia.org/wiki/Estimation_of_covariance_matrices)
- Warning: wiki article uses

$$\mathbf{S} = \sum_{l=0}^{L-1} \mathbf{y}(l) \mathbf{y}^T(l)$$

$$\text{But I use: } \hat{\mathbf{K}}_y = \frac{1}{L} \sum_{l=0}^{L-1} \mathbf{y}(l) \mathbf{y}^H(l)$$

- So our "B" definition differs

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## Spectral Theorem

$$-L \ln \det \mathbf{K}_y - L \text{tr}\left[\hat{\mathbf{K}}_y \mathbf{K}_y^{-1}\right]$$

- Since  $\mathbf{K}_y$  is Hermitian symmetric, we can decompose

$$\hat{\mathbf{K}}_y = \hat{\mathbf{K}}_y^{1/2} \hat{\mathbf{K}}_y^{1/2} \quad \hat{\mathbf{K}}_y^{1/2} = \sum_{i=0}^{M-1} \sqrt{\lambda_i} \mathbf{v}_i \mathbf{v}_i^H$$

$$-L \ln \det \mathbf{K}_y - L \text{tr}\left[\hat{\mathbf{K}}_y^{1/2} \hat{\mathbf{K}}_y^{1/2} \mathbf{K}_y^{-1}\right]$$

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## A Substitution

$$-L \ln \det \hat{\mathbf{K}}_y - L \text{tr} \left[ \hat{\mathbf{K}}_y^{1/2} \hat{\mathbf{K}}_y^{-1} \hat{\mathbf{K}}_y^{1/2} \right]$$

- Let  $\mathbf{B} = \hat{\mathbf{K}}_y^{1/2} \hat{\mathbf{K}}_y^{-1} \hat{\mathbf{K}}_y^{1/2}$

$$\hat{\mathbf{K}}_y^{-1} = \hat{\mathbf{K}}_y^{-1/2} \mathbf{B} \hat{\mathbf{K}}_y^{-1/2}$$

$$\hat{\mathbf{K}}_y = \hat{\mathbf{K}}_y^{1/2} \mathbf{B}^{-1} \hat{\mathbf{K}}_y^{1/2}$$

$$-L \ln \det [\hat{\mathbf{K}}_y^{1/2} \mathbf{B}^{-1} \hat{\mathbf{K}}_y^{1/2}] - L \text{tr} [\mathbf{B}]$$

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## Diagonalization

$$-L \ln \det [\hat{\mathbf{K}}_y^{1/2} \mathbf{B}^{-1} \hat{\mathbf{K}}_y^{1/2}] - L \text{tr} [\mathbf{B}]$$

$$= L \ln \det [\mathbf{B}] - L \text{tr} [\mathbf{B}]$$

- We can diagonalize:  $\mathbf{B} = \sum_{i=0}^{M-1} \beta_i \mathbf{u}_i \mathbf{u}_i^H$

$$L \ln \left[ \prod_{i=0}^{M-1} \beta_i \right] - L \sum_{i=0}^{M-1} \beta_i$$

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## Proof of What We've Always Assumed

$$L \sum_{i=0}^{M-1} \ln \beta_i - L \sum_{i=0}^{M-1} \beta_i$$

- Take derivatives:

$$\frac{1}{\beta_i} - 1 = 0 \Rightarrow \beta_i = 1 \Rightarrow \mathbf{B} = \mathbf{I}$$

$$\hat{\mathbf{K}}_y^{(ML)} = \hat{\mathbf{K}}_y^{1/2} \mathbf{B}^{-1} \hat{\mathbf{K}}_y^{1/2} = \hat{\mathbf{K}}_y$$

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## References

- D. Maiwald and D. Kraus, "Calculation of moments of complex Wishart and complex inverse Wishart distributed matrices," *IEE Proc. Radar, Sonar, and Navigation*, Vol. 147, No. 4, Aug. 2000, pp. 162-168.
- As usual, notation in paper differs from that in slides; be careful

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## Statistics on Empirical Covariances

$$\underline{\mathbf{y}} \sim CN(0, \mathbf{K}_y)$$

$$\underline{\mathbf{S}} = \sum_{l=1}^{L-1} \underline{\mathbf{y}}(l) \underline{\mathbf{y}}^T(l)$$

- The statistic  $\underline{\mathbf{S}}$  is complex Wishart distributed with L degrees of freedom

$$\underline{\mathbf{S}} \sim W(L, \mathbf{K}_y)$$

$$\hat{\mathbf{K}}_y \sim W(L, \frac{1}{L} \mathbf{K}_y)$$

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## Additive Property of Wishart RMs

$$\underline{\mathbf{S}}_A \sim W(L_A, \mathbf{K}_y)$$

$$\underline{\mathbf{S}}_B \sim W(L_B, \mathbf{K}_y)$$

$$\underline{\mathbf{S}}_B + \underline{\mathbf{S}}_B \sim W(L_A + L_B, \mathbf{K}_y)$$

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## Complex Wishart Density (1)

$$p(\mathbf{S}) = \frac{1}{Z} (\det \mathbf{S})^{L-M} \exp[-\text{tr}(\mathbf{K}_y^{-1} \mathbf{S})]$$

$$Z = \pi^{M(M-1)/2} \prod_{m=0}^{M-1} \Gamma(L-m) [\det(\mathbf{K}_y)]^L$$

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## Complex Wishart Density (2)

$$p(\hat{\mathbf{K}}_y) = \frac{1}{Z} (\det \hat{\mathbf{K}}_y)^{L-M} \exp[-L \text{tr}(\mathbf{K}_y^{-1} \hat{\mathbf{K}}_y)]$$

$$Z = \pi^{M(M-1)/2} \prod_{m=0}^{M-1} \Gamma(L-m) \left[ \frac{1}{L^M} \det(\mathbf{K}_y) \right]^L$$

$$= \pi^{M(M-1)/2} \prod_{m=0}^{M-1} \Gamma(L-m) [\det(\mathbf{K}_y)]^L L^{-LM}$$

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## Moments of Wishart Random Matrices

$$E[\hat{\mathbf{K}}_y] = \mathbf{K}_y$$

$$E[(\hat{\mathbf{K}}_y)_{ij}(\hat{\mathbf{K}}_y)_{kl}] = (\mathbf{K}_y)_{ij}(\mathbf{K}_y)_{kl} + \frac{1}{L}(\mathbf{K}_y)_{kj}(\mathbf{K}_y)_{il}$$

$$\text{var}[(\hat{\mathbf{K}}_y)_{ii}] = \frac{1}{L}(\mathbf{K}_y)_{ii}^2$$

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## Visualizing the Moments

$$E[(\hat{\mathbf{K}}_y)_{13}(\hat{\mathbf{K}}_y)_{25}] = (\mathbf{K}_y)_{13}(\mathbf{K}_y)_{25}$$

$$\begin{bmatrix} \cdot & \cdot & 13 & \cdot & 15 \\ \cdot & \cdot & 23 & \cdot & 25 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} + \frac{1}{L}(\mathbf{K}_y)_{23}(\mathbf{K}_y)_{15}$$

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